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LETTER TO THE EDITOR

The velocity dependence of inter-soliton forces

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Abstract. A simple method of calculating the force between extended particles reveals the velocity dependence of inter-soliton forces in the sine-Gordon model.

The only definition of force, the rate of change of momentum, is used herein to find the force between solitons in the sine-Gordon model. This method is more quantitative than that of Rubinstein (1970) and more dynamically revealing than that of Perring and Skyrme (1962), as elaborated by Rajaraman (1977), or that of Troost as extended by Hsu (1980). The method described by Rosen and Rosenstock (1952) will be recapitulated here in the context of the sine-Gordon model.

The familiar, rescaled, Lagrangian density $\mathscr{L}$ for the model is

$$\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+(\cos\phi-1)$$

where $g_{\mu\nu}=\operatorname{diag}(1,-1)$. The solutions of the corresponding wave equation

$$g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\sin\phi=0 \qquad (1)$$

which will be of interest here are as follows.

(i) The single-soliton solution

$$\phi_{\rm S}=2\Pi[-\gamma(x-vt)]$$

where Π is Lobachevskiy's angle of parallelism (Gradshteyn and Ryzhik 1965, p 43), v is the velocity and γ the Lorentz factor $(1-v^2)^{-1/2}$. The antisoliton solution, $\phi_{\bar{\rm S}}$, is the negative of the soliton solution.

(ii) The soliton–soliton scattering solution

$$\phi_{\rm SS}=4\tan^{-1}[v\sinh(\gamma x)/\cosh(v\gamma t)]$$

which has the limiting forms

$$\phi_{\rm SS}\begin{cases}\to\phi_{\rm S}[\gamma(x-\delta-vt)]+\phi_{\rm S}[\gamma(x+\delta+vt)] & \text{as } t\to-\infty,\\ \to\phi_{\rm S}[\gamma(x-\delta+vt)]+\phi_{\rm S}[\gamma(x+\delta-vt)] & \text{as } t\to\infty,\end{cases}$$

where $\delta=(\ln v)/\gamma<0$.

(iii) The soliton–antisoliton scattering solution

$$\phi_{\rm S\bar{S}}=4\tan^{-1}[\sinh(v\gamma t)/v\cosh(\gamma x)]$$

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which has the limiting forms

$$\phi_{S\bar{S}} \begin{cases} \to \phi_{\bar{S}}[\gamma(x-\delta-vt)]+\phi_S[\gamma(x+\delta+vt)] & \text{as } t\to-\infty, \\ \to \phi_S[\gamma(x-\delta+vt)]+\phi_{\bar{S}}[\gamma(x+\delta-vt)] & \text{as } t\to\infty. \end{cases}$$

The bound state of the soliton and antisoliton, the breather mode, is obtained from (iii) by the replacement of v with $\mathrm{i}u$.

To find the inter-soliton force, use will be made of the energy–momentum tensor density

$$T^{\mu}{}_{\nu} = \phi_{,\nu}\partial\mathscr{L}/\partial\phi_{,\mu} - \delta^{\mu}{}_{\nu}\mathscr{L} \qquad \text{where } \phi_{,\nu} = \partial_{\nu}\phi.$$

This is conserved by virtue of the wave equation (1). The momentum in a spatial interval (a, b) is

$$P|_a^b = \int_a^b \mathrm{d}x T^{01}$$

and the force on the spatial interval (rate of change of momentum) is $\partial_0 \int_a^b \mathrm{d}x T^{01} = -T^{11}|_a^b$ by the conservation of the energy–momentum tensor.

To find the force on a soliton, it would be desirable to take an interval of fixed size about the soliton centre, but in general for multisoliton solutions this is inexpedient. (A possibility is to take the interval at half peak height.) To avoid these complications for two solitons we will take centre of mass coordinates. The momentum of one soliton will be $P|_{-\infty}^{0}$ and of the other $P|_0^{\infty} = -P|_{-\infty}^{0}$. The force on the one soliton will be $-T^{11}|_{-\infty}^{0}$ and on the other $-T^{11}|_0^{\infty} = T^{11}|_{-\infty}^{0}$. This interpretation of momentum and force is like the situation of point particle centre of mass scattering. When $P|_{-\infty}^{0}$ is positive this will describe an incoming soliton, and when $P|_{-\infty}^{0}$ is negative an outgoing soliton. Although the force is not prescribed in an explicitly Lorentz-invariant manner, incoming and outgoing, left and right are preserved by Lorentz transformations.

For the soliton–soliton solution (ii) the coordinates are already the centre of mass coordinates, and the momentum

$$P_{SS}|_{-\infty}^{0} = -8v\gamma \tanh(v\gamma t)$$

is incoming for $t<0$ and outgoing for $t>0$. The force

$$F_{SS} = -8v^2\gamma^2/\cosh^2(v\gamma t),$$

which is repulsive for all t, has a limiting form

$$F_{SS} \to -8v^2\gamma^2 \exp(-2v\gamma|t|) \qquad \text{as } |t|\to\infty.$$

It is clear that the above procedure gives the force as a function of time and not the ill defined soliton position, but as $|t|\to\infty$ the soliton separation d becomes distinct, $d\to 2(v|t|-\delta)$, and so

$$F_{SS} \to -8\gamma^2 \exp(-\gamma d) \qquad \text{as } d\to\infty,$$

showing the velocity dependence of the limiting form of the force. As $v\to 0$ the force tends to $-8\exp(-d)$, which apart from the numerical factor agrees with the static behaviour found by previous authors.

For the soliton–antisoliton solution (iii) (again already in centre of mass)

$$P_{S\bar{S}}|_{-\infty}^{0} = -4v\gamma \sinh(2v\gamma t)/[\sinh^2(v\gamma t)+v^2]$$

is incoming for $t<0$ and outgoing for $t>0$. The force is more complicated than for soliton–soliton scattering:

$$F_{s\bar{s}} = 8v^2\gamma^2[\sinh^2(v\gamma t) - v^2\cosh(2v\gamma t)]/[\sinh^2(v\gamma t)+v^2]^2,$$

for which the limiting form

$$F_{s\bar{s}} \to 8\exp(-\gamma d) \qquad \text{as } d\to\infty$$

is attractive.

It can easily be seen that the half momentum and the force for the breather mode are, as would be expected, oscillatory.

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